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THE DIRECT METHOD OF RESEARCH OF THE OSCILLATION PROCESSES FOR THE WAVE EQUATION WITH THE PIECEWISE CONTINUOUS DISTRIBUTION PARAMETERS

A new solving scheme of the general first boundary value problem for a hyperbolic type equation with piecewise continuous coefficients and stationary heterogeneous was proposed and justified. In the basis of the solving scheme these is a concept of quasi-derivatives, a modern theory of systems of linear differential equations, the classical Fourier method and a reduction method. The advantage of this method lies in a possibility to examine a problem on each breakdown segment and then to combine obtained solutions on the basis of matrix calculation. This approach allows to use software tools for the solution.

Key words: kvazidifferential equation, the boundary value problem, the Cauchy matrix, the eigenvalues problem, the method of Fourier and the method of eigenfunctions.

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ПРЯМИЙ МЕТОД ДОСЛІДЖЕННЯ КОЛИВНИХ ПРОЦЕСІВ ДЛЯ ХВИЛЬОВОГО РІВНЯННЯ З КУСКОВО-НЕПЕРЕРВНИМ РОЗПОДІЛОМ ПАРАМЕТРІВ

Запропоновано та обґрунтовано нову схему розв’язування загальної першої крайової задачі для рівняння гіперболічного типу з кусково-неперервними коефіцієнтами та стаціонарною неоднорідністю. В основу схеми розв’язування покладено концепцію квазіпохідних, сучасну теорію систем лінійних диференціальних рівнянь, а також класичний метод Фур’є та метод редукції. Перевагою методу є можливість розглянути задачу на кожному відрізку розбиття, а потім на основі матричного числення об’єднати отримані розв’язки. Такий підхід дозволяє застосувати програмні засоби до процесу вирішення задачі та графічної ілюстрації розв’язку.

Ключові слова: квазідиференціальне рівняння, крайова задача, матриця Коші, задача на власні значення, метод Фур’є та метод власних функцій.

Introduction

The main methods for solving nonstationary boundary value problems are the separation of variables method, Green’s function method, method of integral transforms, approximate methods and numerical methods.

The scheme proposed in this article belongs to the direct methods for solving boundary value problems for hyperbolic type equations. In the basis of the solving scheme these is a concept of quasi-derivatives [5] that allows to by pass the problem of multiplication of generalized functions.

First of all, a mixed problem for the heat equation with piecewise continuous coefficients by the general boundary conditions of the first kind [6] was solved.

This article examines the general first boundary value problem for a hyperbolic type equation with piecewise continuous coefficients and stationary heterogeneous. The usage of the reduction method for solving this problem reduces the possibility to find the solution of the stationary inhomogeneous boundary value problem with the initial boundary conditions and the mixed problem with the zero boundary conditions for an inhomogeneous equation.

I. Results and discussion.

Let $0 = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < x_{i+1} \ldots < x_{n-1} < x_n = l$ – arbitrary partition of the segment $[0;l]$ of the real axis Ox into n parts.

Let's declare the main designations:

θ_i – characteristic function of the interval $\left[x_i;x_{i+1}\right)$, that is $\theta_i\left(x\right) = \begin{cases} 1, x \in \left[x_i, x_{i+1}\right), \\ 0, x \notin \left[x_i, x_{i+1}\right), \end{cases}$ $i = \overline{0, n-1}.$

Remark 1. *If* $a_1 = \sum_{i=0}^{n-1} a_{1i}\theta_i$, $a_2 = \sum_{i=0}^{n-1} a_{2i}\theta_i$, *then* $a_1 \cdot a_2 = \sum_{i=0}^{n-1} a_{1i} \cdot a_{2i}\theta_i$. *In particular, if* $a = \sum_{i=0}^{n-1} a_i\theta_i$, *then* $\frac{1}{a} = \sum_{i=0}^{n-1} a_i^{-1}\theta_i$.

Let

$r(x) = \sum_{i=0}^{n-1} r_i(x)\theta_i$, $r_i(x) \in C[x_i;x_{i+1})$, $r_i(x) > 0$;

$\lambda(x) = \sum_{i=0}^{n-1} \lambda_i(x)\theta_i$, $\lambda_i(x) \in C[x_i;x_{i+1})$, $\lambda_i(x) > 0$;

$f(x) = g(x) + s(x) = \sum_{i=0}^{n-1} g_i(x)\theta_i + \sum_{i=1}^{n-1} s_i\delta_i(x - x_i)$, where $g_i(x)$ –function on the interval $[x_i;x_{i+1})$, s_i – real numbers, $\delta_i = \delta_i(x - x_i)$ – δ - Dirac's function with a carrier at the point $x = x_i$.

Let's examine the general first boundary value problem for a hyperbolic type equation

$$r(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial u}{\partial x}\right) + f(x),\ x \in (x_0;x_n),\ t \in (0;+\infty), \tag{1}$$

with the boundary conditions

$$\begin{cases} u(x_0,t) = \psi_0(t), \\ u(x_n,t) = \psi_n(t), \end{cases} t \in [0;+\infty) \tag{2}$$

and the initial conditions

$$\begin{cases} u(x,0) = \varphi_0(x), \\ \dfrac{\partial u}{\partial t}(x,0) = \varphi_1(x), \end{cases} x \in [x_0;x_n], \tag{3}$$

where $\psi_0(t)$, $\psi_n(t) \in C^2(0;+\infty)$, $\varphi_0(x)$, $\varphi_1(x)$ – are piecewise continuous on $(x_0;x_n)$.

The method of reduction for finding a solution of the problem is described in detail in [1, 7, 8]. In accordance with this method we can find a solution of the problem as a sum of two functions

$$u(x,t) = w(x,t) + v(x,t). \tag{4}$$

II. Building the function $w(x,t)$.

Let's define a function $w(x,t)$ as a solution of a boundary value problem

$$(\lambda w_x')_x' = -f(x), \tag{5}$$

$$\begin{cases} w(x_0,t)=\psi_0(t), \\ w(x_n,t)=\psi_n(t), \end{cases} t\in[0;+\infty). \qquad (6)$$

In the basis of the solving method of the problem (5), (6) is the concept of quasi-derivatives [4].

Let's introduce the vectors $\overline{W}=\begin{pmatrix} w \\ w^{[1]} \end{pmatrix}$, where $w^{[1]}=\lambda w_x{}'$, $\overline{G}=\begin{pmatrix} 0 \\ -g(x) \end{pmatrix}$, $\overline{S}_i=\begin{pmatrix} 0 \\ -s_i \end{pmatrix}$, $\overline{S}=\sum_{i=1}^{n-1}\overline{S}_i\cdot\delta_i$. Using these definitions, the quasi-differential equation (5) reduces to the equivalent system of differential equations of the first order

$$\overline{W}_x{}'=\begin{pmatrix} 0 & \frac{1}{\lambda} \\ 0 & 0 \end{pmatrix}\overline{W}+\overline{G}+\overline{S}. \qquad (7)$$

As a solution of the system (7) we take a vector function $\overline{W}(x,t)$ that fulfils the system (7) almost everywhere.

Boundary conditions (6) can be written down in vector form

$$P\cdot\overline{W}(x_0,t)+Q\cdot\overline{W}(x_n,t)=\overline{\Gamma}(t), \qquad (8)$$

where $P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\overline{\Gamma}(t)=\begin{pmatrix} \psi_0(t) \\ \psi_n(t) \end{pmatrix}$.

Let $w_i(x,t)$, $w_i^{[1]}(x,t)$ and $g_i(x)$ be defined on the interval $[x_i;x_{i+1})$. Let's define

$$w(x,t)=\sum_{i=0}^{n-1}w_i(x,t)\theta_i. \qquad (9)$$

On the interval $[x_i;x_{i+1})$ the system (7) is represented as

$$\begin{pmatrix} w_i \\ w_i^{[1]} \end{pmatrix}_x^{\prime}=\begin{pmatrix} 0 & \frac{1}{\lambda_i(x)} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_i \\ w_i^{[1]} \end{pmatrix}+\begin{pmatrix} 0 \\ -g_i(x) \end{pmatrix}+\begin{pmatrix} 0 \\ -s_i \end{pmatrix}, \qquad (10)$$

where $s_0 \overset{def}{=} 0$.

Let's examine a homogeneous system that corresponds to the system (10)

$$\begin{pmatrix} w_i \\ w_i^{[1]} \end{pmatrix}'=\begin{pmatrix} 0 & \frac{1}{\lambda_i(x)} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_i \\ w_i^{[1]} \end{pmatrix}.$$

The Cauchy matrix $B_i(x,s)$ of such a system is represented as

$$B_i(x,s)=\begin{pmatrix} 1 & b_i(x,s) \\ 0 & 1 \end{pmatrix}, \text{ where } b_i(x,s)=\int_s^x\frac{dz}{\lambda_i(z)} \text{ (see [3]).} \qquad (11)$$

Let's define (for an arbitrary $k\geq i$)

$$B(x_k,x_i) \overset{def}{=} B_{k-1}(x_k,x_{k-1})\cdot B_{k-2}(x_{k-1},x_{k-2})\cdot\ldots\cdot B_i(x_{i+1},x_i). \qquad (12)$$

The structure (11) of the matrices $B_i(x,s)$ allows us to define the structure of the matrix (12)

$$B(x_k, x_i) = \begin{pmatrix} 1 & \sum_{m=i}^{k-1} b_m(x_{m+1}, x_m) \\ 0 & 1 \end{pmatrix},$$

besides that $B(x_k, x_k) \overset{def}{=} E$, where E is an identity matrix.

The solution of the system (10) on the interval $[x_i; x_{i+1})$ is

$$\overline{W}_i(x,t) = B_i(x, x_i)\cdot \overline{P}_i + \int_{x_i}^{x} B_i(x,s)\cdot \overline{G}_i(s)\,ds, \tag{13}$$

where $\overline{P}_i$ is a yet unknown vector.

Similarly on the interval $[x_{i-1}; x_i)$

$$\overline{W}_{i-1}(x,t) = B_{i-1}(x, x_{i-1})\cdot \overline{P}_{i-1} + \int_{x_{i-1}}^{x} B_{i-1}(x,s)\cdot \overline{G}_{i-1}(s)\,ds. \tag{14}$$

At the point $x = x_i$ the conjugation condition has to be fulfilled that is $\overline{W}_i(x_i,t) = \overline{W}_{i-1}(x_i,t) + \overline{S}_i$ [9]. As a result we get a recurrence relation

$$\overline{P}_i = B_{i-1}(x_i, x_{i-1})\cdot \overline{P}_{i-1} + \int_{x_{i-1}}^{x_i} B_{i-1}(x_i,s)\cdot \overline{G}_{i-1}(s)\,ds + \overline{S}_i. \tag{15}$$

By the method of mathematical induction from (15) the following is received

$$\overline{P}_i = B(x_i, x_0)\cdot \overline{P}_0 + \sum_{k=0}^{i} B(x_i, x_k)\overline{Z}_k, \tag{16}$$

where $\overline{Z}_k = \int_{x_{k-1}}^{x_k} B_{k-1}(x_k,s)\cdot \overline{G}_{k-1}(s)\,ds + \overline{S}_k$, $k = \overline{1, n-1}$, note that $\overline{Z}_0 \overset{def}{=} \overline{0}$, $\overline{S}_n \overset{def}{=} \overline{0}$;

$\overline{P}_0$ – is the initial (unknown) vector.

In order to find $\overline{P}_0$ the boundary conditions (8) should be used, where we define

$\overline{W}(x_0,t) \overset{def}{=} \overline{P}_0$,

$$\overline{W}(x_n,t) \overset{def}{=} \overline{W}_{n-1}(x_n,t) = B_{n-1}(x_n, x_{n-1})\overline{P}_{n-1} + \int_{x_{n-1}}^{x_n} B_{n-1}(x_n,s)\cdot \overline{G}_{n-1}(s)\,ds =$$

$$= B(x_n, x_0)\overline{P}_0 + \sum_{k=0}^{n} B(x_n, x_k)\overline{Z}_k.$$

Then $[P + QB(x_n, x_0)]\overline{P}_0 + Q\sum_{k=0}^{n} B(x_n, x_k)\overline{Z}_k = \overline{\Gamma}$, and as a result

$$\overline{P}_0 = [P + Q\cdot B(x_n, x_0)]^{-1}\cdot\left(\overline{\Gamma} - Q\sum_{k=0}^{n} B(x_n, x_k)\overline{Z}_k\right). \tag{17}$$

Let's evaluate

$$[P+Q\cdot B(x_n,x_0)]^{-1}=\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}1&\sum\limits_{m=0}^{n-1}b_m(x_{m+1},x_m)\\0&1\end{pmatrix}\right]^{-1}=\begin{pmatrix}1&0\\-\dfrac{1}{\sigma_n}&\dfrac{1}{\sigma_n}\end{pmatrix},$$

where $\sigma_n=\sum\limits_{m=0}^{n-1}b_m(x_{m+1},x_m)$, $\sigma_0\overset{def}{=}0$;

$$\overline{\Gamma}-Q\sum_{k=0}^{n}B(x_n,x_k)\overline{Z}_k=\begin{pmatrix}\psi_0(t)\\\psi_n(t)\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\sum_{k=0}^{n}B(x_n,x_k)\left(\int_{x_{k-1}}^{x_k}B_{k-1}(x_k,s)\cdot\overline{G}_{k-1}(s)\,ds+\overline{S}_k\right). \quad (18)$$

Let's write down the right side part (18) in a matrix form

$$\int_{x_{k-1}}^{x_k}B_{k-1}(x_k,s)\cdot\overline{G}_{k-1}(s)\,ds+\overline{S}_k=\int_{x_{k-1}}^{x_k}\begin{pmatrix}1&b_{k-1}(x_k,s)\\0&1\end{pmatrix}\cdot\begin{pmatrix}0\\-g_{k-1}(s)\end{pmatrix}ds+\begin{pmatrix}0\\-s_k\end{pmatrix}=$$

$$=\begin{pmatrix}-\int\limits_{x_{k-1}}^{x_k}b_{k-1}(x_k,s)\cdot g_{k-1}(s)\,ds\\-\int\limits_{x_{k-1}}^{x_k}g_{k-1}(s)\,ds-s_k\end{pmatrix}\overset{def}{=}\begin{pmatrix}I_{k-1}(x_k)\\I_{k-1}^{[1]}(x_k)-s_k\end{pmatrix}=\overline{Z}_k;$$

$$\sum_{k=0}^{n}B(x_n,x_k)\begin{pmatrix}I_{k-1}(x_k)\\I_{k-1}^{[1]}(x_k)-s_k\end{pmatrix}=\sum_{k=0}^{n}\begin{pmatrix}1&\sum\limits_{m=k}^{n-1}b_m(x_{m+1},x_m)\\0&1\end{pmatrix}\begin{pmatrix}I_{k-1}(x_k)\\I_{k-1}^{[1]}(x_k)-s_k\end{pmatrix}=$$

$$=\begin{pmatrix}\sum\limits_{k=0}^{n}\left(I_{k-1}(x_k)+(I_{k-1}^{[1]}(x_k)-s_k)\cdot\sum\limits_{m=k}^{n-1}b_m(x_{m+1},x_m)\right)\\\sum\limits_{k=0}^{n}\left(I_{k-1}^{[1]}(x_k)-s_k\right)\end{pmatrix}.$$

Thus, we receive

$$\overline{\Gamma}-Q\sum_{k=0}^{n}B(x_n,x_k)\overline{Z}_k=$$

$$=\begin{pmatrix}\psi_0(t)\\\psi_n(t)-\sum\limits_{k=0}^{n}\left(I_{k-1}(x_k)+(I_{k-1}^{[1]}(x_k)-s_k)\cdot\sum\limits_{m=k}^{n-1}b_m(x_{m+1},x_m)\right)\end{pmatrix}, \quad (19)$$

Let's substitute (19) to (17)

$$\overline{P}_0=\begin{pmatrix}\psi_0(t)\\\dfrac{\psi_n(t)-\psi_0(t)}{\sigma_n}-\dfrac{1}{\sigma_n}\sum\limits_{k=0}^{n}\left(I_{k-1}(x_k)+(I_{k-1}^{[1]}(x_k)-s_k)\cdot\sum\limits_{m=k}^{n-1}b_m(x_{m+1},x_m)\right)\end{pmatrix}. \quad (20)$$

Based on the formulas (13), (16), (20), after performed transformations an image of the vector function $\overline{W}_i(x,t)$ on the interval $[x_i;x_{i+1})$ is received

$$\overline{W}_i(x,t) = B_i(x,x_i)\cdot\left(B(x_i,x_0)\cdot\overline{P}_0 + \sum_{k=0}^{i} B(x_i,x_k)\overline{Z}_k \right) + \int_{x_i}^{x} B_i(x,s)\cdot\overline{G}_i(s)\,ds =$$

$$= \begin{pmatrix} 1 & b_i(x,x_i)+\sigma_i \\ 0 & 1 \end{pmatrix}\cdot\overline{P}_0 +$$

$$+\begin{pmatrix} \sum\limits_{k=0}^{i}\left(I_{k-1}(x_k) + (I_{k-1}^{[1]}(x_k) - s_k)\sum\limits_{m=k}^{i-1} b_m(x_{m+1},x_m) \right) + b_i(x,x_i)\sum\limits_{k=0}^{i}\left(I_{k-1}^{[1]}(x_k) - s_k \right) + I_i(x) \\ \sum\limits_{k=0}^{i}\left(I_{k-1}^{[1]}(x_k) - s_k \right) + I_i^{[1]}(x) \end{pmatrix}. \quad (21)$$

The first coordinate of the vector $\overline{W}_i(x,t)$ in (21) is indeed the searched function $w_i(x,t)$. Therefore

$$w_i(x,t) = \psi_0(t) + \left(b_i(x,x_i)+\sigma_i\right)\cdot\frac{\psi_n(t)-\psi_0(t)}{\sigma_n} - \frac{1}{\sigma_n}\left(b_i(x,x_i)+\sigma_i\right)\times$$

$$\times\sum_{k=0}^{n}\left(I_{k-1}(x_k) + (I_{k-1}^{[1]}(x_k) - s_k)\sum_{m=k}^{n-1} b_m(x_{m+1},x_m) \right) +$$

$$+\sum_{k=0}^{i}\left(I_{k-1}(x_k) + (I_{k-1}^{[1]}(x_k) - s_k)\sum_{m=k}^{i-1} b_m(x_{m+1},x_m) \right) + b_i(x,x_i)\sum_{k=0}^{i}\left(I_{k-1}^{[1]}(x_k) - s_k \right) + I_i(x). \quad (22)$$

By substituting the expression (22) into (9), the solution on the whole interval $[x_0;x_n]$ is received.

III. Building the function $v(x,t)$.

Let's write down a mixed problem for the function $v(x,t)$. Substituting (4) into (1) and considering that the function $w(x,t)$ fulfills (5), an inhomogeneous equation is received

$$r(x)\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x} \right) = -r(x)\frac{\partial^2 w}{\partial t^2},\ x\in(x_0;x_n),\ t\in(0;+\infty). \quad (23)$$

Let's substitute (4) into the initial conditions (3). Initial conditions for the function $v(x,t)$ are received

$$\begin{cases} v(x,0) = \varphi_0(x) - w(x,0) \overset{def}{=} \Phi_0(x), \\ \dfrac{\partial v}{\partial t}(x,0) = \varphi_1(x) - \dfrac{\partial w}{\partial t}(x,0) \overset{def}{=} \Phi_1(x), \end{cases} \quad x\in[x_0;x_n]. \quad (24)$$

Since the function $w(x,t)$ fulfills the boundary conditions (6), than from (4) the boundary conditions for the function $v(x,t)$ will be the following.

$$\begin{cases} v(x_0,t) = 0, \\ v(x_n,t) = 0, \end{cases} \quad t\in[0;+\infty). \quad (25)$$

Therefore under the condition that the solution $w(x,t)$ of the problem (5), (6) is known, the function $v(x,t)$ is the solution of the mixed problem (23)-(25).

IV. The Fourier method and the eigenvalue problem.

1. Expansion by eigenfunctions.

Let's find the solution of corresponding homogeneous equation for the equation (23) однорідного рівняння

$$r(x)\frac{\partial^2 v}{\partial t^2}=\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x}\right), \tag{26}$$

that fulfills boundary conditions (25). Now let's find its nontrivial solution

$$v(x,t)=T(t)\cdot X(x), \tag{27}$$

where $T(t)$, $X(x)$ – are yet unknown functions [1].

By substituting (27) into (26) and executing division both of parts (26) on $r(x)\cdot T(t)\cdot X(x)$ the quasi-differential equation is received

$$\left(\lambda(x)X'(x)\right)'+\omega^2 r(x)X(x)=0, \tag{28}$$

where ω is a parameter.

Let's substitute (27) into the conditions (25) and take into account that $T(t)\neq 0$. The following boundary conditions are received

$$\begin{aligned} X(x_0)&=0,\\ X(x_n)&=0. \end{aligned} \tag{29}$$

The problem (28), (29) is the eigenvalue problem. The properties of the eigenvalues ω_k and the eigenfunctions $X_k(x,\omega_k)$ of the problem (28), (29) are described in detail in [8]. The expansion by the eigenfunctions $X_k(x,\omega_k)$ of function $F(x)$ which fulfills some conditions is the following

$$F(x)=\sum_{k=1}^{\infty}F_k\cdot X_k(x,\omega_k), \tag{30}$$

where the Fourier coefficients F_k are computed by the formulas

$$F_k=\frac{1}{\|X_k\|^2}\cdot\int_{x_0}^{x_n}F(x)\cdot X_k(x,\omega_k)\cdot r(x)\,dx. \tag{31}$$

Let's note, that

$$\|X_k\|^2=\int_{x_0}^{x_n}X_k^2(x,\omega_k)\cdot r(x)\,dx. \tag{32}$$

Let's define conditions, which fulfill function $F(x)$. Let's consider, that $F(x)$ is an absolutely continuous function that has different analytical expressions on each of the intervals $\left[x_i;x_{i+1}\right)$, which allows the image

$$F(x)=\sum_{i=0}^{n-1}F_i(x)\cdot\theta_i \tag{33}$$

on the interval $[x_0;x_n]$.

Functions of the type (33) are piecewise-continuous functions with the gaps of the first-type in the points x_i, $i=\overline{0,n-1}$. Functions of the type (33) are added, multiplied and integrated the following way:

if $F_1(x)=\sum_{i=0}^{n-1} F_{1i}(x)\cdot\theta_i$, $F_2(x)=\sum_{i=0}^{n-1} F_{2i}(x)\cdot\theta_i$, then

$$F_1 \pm F_2 = \sum_{i=0}^{n-1}\left(F_{1i} \pm F_{2i}\right)\theta_i , \quad F_1 \cdot F_2 = \sum_{i=0}^{n-1}\left(F_{1i}\cdot F_{2i}\right)\cdot\theta_i ,$$

$$\int_{x_0}^{x_n} F_1(x)\cdot F_2(x)\cdot r(x)\,dx = \sum_{i=0}^{n-1}\int_{x_0}^{x_n} F_{1i}(x)\cdot F_{2i}(x)\cdot r_i(x)\,dx , \tag{34}$$

$$\left\|F_k\right\|^2 = \int_{x_0}^{x_n} F_k^2(x)\cdot r(x)\,dx = \sum_{i=0}^{n-1}\int_{x_0}^{x_n} F_{k\,i}^2(x)\cdot r_i(x)\,dx ,\ k \ge 1 . \tag{35}$$

The expression (34) is the dot product of the functions $F_1(x)$ and $F_2(x)$. The expression (35) is the norm square of the function $F_k(x)$ with the weight $r(x)$.

Let's define

$$X_k(x,\omega_k) = \sum_{i=0}^{n-1} X_{k\,i}(x,\omega_k)\cdot\theta_i . \tag{36}$$

Then for the Fourier coefficients F_k and for the norms squares of functions $X_k(x)$ from the (31) and (32) the following is received

$$F_k = \frac{1}{\left\|X_k\right\|^2}\cdot\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} F_i(x)\cdot X_{k\,i}(x,\omega_k)\cdot r_i(x)\,dx ,$$

$$\left\|X_k\right\|^2 = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} X_{k\,i}^2(x,\omega_k)\cdot r_i(x)\,dx .$$

2. Constructional approach to building eigenfunctions.

Let's introduce a quasi-derivative $X^{[1]} \overset{def}{=} \lambda X'$, a vector $\overline{X} = \begin{pmatrix} X \\ X^{[1]} \end{pmatrix}$ and matrices $A(x) = \begin{pmatrix} 0 & \dfrac{1}{\lambda} \\ -\omega^2 r & 0 \end{pmatrix}$. Now let's reduce a quasi-differential equation (28) to the system of the first order differential equations

$$\overline{X}' = A\overline{X} . \tag{37}$$

The boundary conditions (29) are the next

$$P\overline{X}(x_0) + Q\overline{X}(x_n) = \overline{0} . \tag{38}$$

Let's write down the corresponding system to the system (37) on the interval $\left[x_i, x_{i+1}\right)$ in a following way

$$\overline{X}'_i = A_i\cdot\overline{X}_i ,\ i = \overline{0, n-1} , \tag{39}$$

where $A_i(x) = \begin{pmatrix} 0 & \dfrac{1}{\lambda_i} \\ -\omega^2 r_i & 0 \end{pmatrix}$.

The Cauchy matrix of the system (39) has the following structure $\tilde{B}_i(x,s,\omega)$

$$\tilde{B}(x_i,x_0,\omega) \overset{def}{=} \prod_{j=0}^{i} \tilde{B}_{i-j}(x_{i-j+1},x_{i-j},\omega).$$

Let's define the analog of Cauchy matrix on the whole interval $[x_0;x_n]$

$$\tilde{B}(x,x_0,\omega) \overset{def}{=} \sum_{i=0}^{n-1} \tilde{B}_i(x,x_i,\omega)\cdot\tilde{B}(x_i,x_0,\omega)\cdot\theta_i; \tag{40}$$

$$\tilde{B}(x_n,x_0,\omega) \overset{def}{=} \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) \end{pmatrix}. \tag{41}$$

The nontrivial solution $\overline{X}(x,\omega)$ of the system (37) can be found as

$$\overline{X}(x,\omega) = \tilde{B}(x,x_0,\omega)\cdot\overline{C}, \tag{42}$$

where $\overline{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ is some nonzero vector.

The vector function $\overline{X}(x,\omega)$ has to fulfill the boundary conditions (38). That is

$$P\cdot\overline{X}(x_0,\omega) + Q\cdot\overline{X}(x_n,\omega) = \overline{0},$$

$$\left[P\cdot\tilde{B}(x_0,x_0,\omega) + Q\cdot\tilde{B}(x_n,x_0,\omega)\right]\cdot\overline{C} = \overline{0}.$$

Taking into consideration that $\tilde{B}(x_0,x_0,\omega) = E$, the following equation is received

$$\left[P + Q\cdot\tilde{B}(x_n,x_0,\omega)\right]\cdot\overline{C} = \overline{0}. \tag{43}$$

The nonzero vector $\overline{C}$ exists in (43) if the validity of the following condition is necessary and sufficient

$$\det\left[P + Q\cdot\tilde{B}(x_n,x_0,\omega)\right] = 0. \tag{44}$$

Let's concretize the left part of the characteristic equation (44), taking into consideration the matrices *P*, *Q* and (41)

$$\det\left[P + Q\cdot\tilde{B}(x_n,x_0,\omega)\right] = \det\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) \end{pmatrix}\right] =$$

$$= \det\left[\begin{pmatrix} 1 & 0 \\ b_{11}(\omega) & b_{12}(\omega) \end{pmatrix}\right] = b_{12}(\omega).$$

Let's make the following statement.

Statement 1. *Characteristic equation of the eigenvalue problem (28), (29) is the following*

$$b_{12}(\omega) = 0. \tag{45}$$

As known [8], the roots ω_k of the characteristic equation (45), that are also eigenvalues of the problem (28), (29), are positive and different.

In order to find the nonzero vector $\overline{C}$ let's substitute ω_k with ω into the equation (43). Then the following vectorial equality is received

$$\begin{pmatrix} 1 & 0 \\ b_{11}(\omega_k) & b_{12}(\omega_k) \end{pmatrix}\cdot\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is equivalent to the system of equations

$$\begin{cases} C_1 = 0, \\ b_{11}(\omega_k) \cdot C_1 + b_{12}(\omega_k) \cdot C_2 = 0. \end{cases} \tag{46}$$

Since the determinant of this system $b_{12}(\omega) = 0$, then the system (46) has the following solutions $C_1 = 0$, $C_2 \in \mathbb{R} \setminus \{0\}$. By introducing, for example $C_2 = 1$, $\overline{C} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is received. Let $\overline{X}_k(x, \omega_k)$ be a nontrivial eigenvector that corresponds to the value of ω_k.

Statement 2. *The eigenvectors of the system of differential equations (37) with boundary conditions (38) have the following structure*

$$\overline{X}_k(x, \omega_k) = \tilde{B}(x, x_0, \omega_k) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ k \in \mathbb{N}.$$

Consequence. *The eigenfunctions* $X_k(x, \omega_k)$ *as the first coordinates of the eigenvectors* $\overline{X}_k(x, \omega_k)$ *can be written down as*

$$X_k(x, \omega_k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \tilde{B}(x, x_0, \omega_k) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ k = 1, 2, 3, \ldots. \tag{47}$$

In particular, since the $X_k(x, \omega_k)$ is (36), then from (40) and (47) follows that

$$X_{k\,i}(x, \omega_k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \tilde{B}_i(x, x_i, \omega_k) \cdot \tilde{B}(x_i, x_0, \omega_k) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ i = \overline{0, n-1}. \tag{48}$$

V. Building a solution $v(x,t)$ to the mixed problem (23) - (25).

In order to solve the problem (23) - (25) let's apply the eigenfunctions method [6], what means that the problem's solution (23) - (25) can be found in a following form

$$v(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot X_k(x, \omega_k), \tag{49}$$

where $T_k(t)$ are unknown functions that will be later defined.

Since $\dfrac{\partial^2 w}{\partial t^2}$ is in the right side of equation (23) let's expand it into the Fourier series by the eigenfunctions $X_k(x, \omega_k)$ of the boundary problem (28), (29)

$$\frac{\partial^2 w}{\partial t^2} = \sum_{k=1}^{\infty} w_k(t) \cdot X_k(x, \omega_k). \tag{50}$$

By substituting the expression (49) into (23) and taking into account (50), the following equality is received

$$r(x) \cdot \sum_{k=1}^{\infty} T_k''(t) \cdot X_k(x, \omega_k) = \sum_{k=1}^{\infty} T_k(t) \cdot \left(\lambda X_k'(x, \omega_k)\right)' - r(x) \cdot \sum_{k=1}^{\infty} w_k(t) \cdot X_k(x, \omega_k).$$

Considering that the eigenfunctions $X_k(x, \omega_k)$ satisfy the equation (28), we get an equality

$$r(x) \cdot \sum_{k=1}^{\infty} T_k''(t) \cdot X_k(x, \omega_k) = -r(x) \sum_{k=1}^{\infty} \omega_k^2 \cdot X_k(x, \omega_k) T_k(t) - r(x) \cdot \sum_{k=1}^{\infty} w_k(t) \cdot X_k(x, \omega_k).$$

Let's divide on $r(x) > 0$ previous equality. We received

$$\sum_{k=1}^{\infty}\left[T_k''(t)+\omega_k^2\cdot T_k(t)+w_k(t)\right]\cdot X_k(x,\omega_k)=0\,. \tag{51}$$

By equating the Fourier coefficients (51) to the zero the differential equations are received

$$T_k''(t)+\omega_k^2\cdot T_k(t)=-w_k(t)\,,\ k=1,2,3,\ldots. \tag{52}$$

The general solution of each of the differential equations (52) is

$$T_k(t)=a_k\cos\omega_k t+d_k\sin\omega_k t-\frac{1}{\omega_k}\int_0^t\sin\omega_k(t-s)\cdot w_k(s)\,ds\,, \tag{53}$$

where a_k, d_k are unknown constants [2].

Let's declare $I(t)=\frac{1}{\omega_k}\int_0^t\sin\omega_k(t-s)\cdot w_k(s)\,ds$. Note that $I(0)=0$, $I_t'(0)=0$.

In order to find the constants a_k, d_k let's expand the right parts of the initial conditions (24) into the Fourier series by the eigenfunctions $X_k(x,\omega_k)$

$$\Phi_0(x)=\sum_{k=1}^{\infty}\Phi_{0\,k}\cdot X_k(x,\omega_k)\,, \tag{54}$$

$$\Phi_1(x)=\sum_{k=1}^{\infty}\Phi_{1\,k}\cdot X_k(x,\omega_k)\,, \tag{55}$$

where $\Phi_{0\,k}$, $\Phi_{1\,k}$ are the corresponding Fourier coefficients.

From (53) follows that

$$T_k(0)=a_k\,, \tag{56}$$

$$T_k'(t)=-a_k\omega_k\sin\omega_k t+d_k\omega_k\cos\omega_k t-I_t'(t)\,,$$

so

$$T_k'(0)=d_k\omega_k\,. \tag{57}$$

From (49), (54) and the first condition in (24) the following is received

$\sum_{k=1}^{\infty}T_k(0)\cdot X_k(x,\omega_k)=\sum_{k=1}^{\infty}\Phi_{0\,k}\cdot X_k(x,\omega_k)$. Now using (56) we receive $T_k(0)=a_k=\Phi_{0\,k}$.

Analogically, from (49), (55) and the second condition in (24)

$\sum_{k=1}^{\infty}T_k'(0)\cdot X_k(x,\omega_k)=\sum_{k=1}^{\infty}\Phi_{1\,k}\cdot X_k(x,\omega_k)$ is received. Using (57) we find $T_k'(0)=d_k\omega_k=\Phi_{1\,k}$, or

$d_k=\frac{\Phi_{1\,k}}{\omega_k}$.

Thus, a solution of the mixed problem (23) - (25) is received in a form of the series

$$v(x,t)=\sum_{k=1}^{\infty}\left[\Phi_{0\,k}\cos\omega_k t+\frac{\Phi_{1\,k}}{\omega_k}\sin\omega_k t-\frac{1}{\omega_k}\int_0^t\sin\omega_k(t-s)\cdot w_k(s)\,ds\right]\cdot X_k(x,\omega_k)\,.$$

Considering (36) and that $v(x,t)=\sum_{i=0}^{n-1}v_i(x,t)\cdot\theta_i$, where $v_i(x,t)$ are defined on the interval $\left[x_i;x_{i+1}\right)$, we receive

$$v_i(x,t)=\sum_{k=1}^{\infty}\left[\Phi_{0\,k}\cos\omega_k t+\frac{\Phi_{1\,k}}{\omega_k}\sin\omega_k t-\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-s)\cdot w_k(s)\,ds\right]\cdot X_{k\,i}(x,\omega_k), \qquad (58)$$

where the functions $X_{ki}(x,\omega_k)$ are computed by the formula (48).

Considering (22), (58), the solution of the problem (1) - (3) is received

$$u(x,t)=\sum_{i=0}^{n-1}\left[w_i(x,t)+v_i(x,t)\right]\cdot\theta_i .$$

Conclusion.

The theorem about the expansion by the eigenfunctions is adapted for the case of differential equations with piecewise constant (by the spatial variable) coefficients.

Explicit formulas for finding the solution and its quasi-derivatives for any partial interval of the main interval that are valid for arbitrary finite numbers of the first type break points of the earlier referred coefficients are received.

This scheme of problem examination was considered in a case of rectangular Cartesian coordinate system. However, it remains valid in a case of any curvilinear orthogonal coordinates. The advantage of this method lies in the possibility to examine a problem on each breakdown segment and then to combine obtained solutions on the basis of matrix calculation. This approach allows the use of software tools for solving the problem. The received results have a direct application to applied problems.

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